

Refined Topological Vertex and Instanton Counting

Masato Taki

*Department of Physics, Faculty of Science, University of Tokyo,
Bunkyo-ku, Tokyo 113-0033, Japan.*

tachyon@hep-th.phys.s.u-tokyo.ac.jp

Abstract

It has been proposed recently that topological A-model string amplitudes for toric Calabi-Yau 3-folds in non self-dual graviphoton background can be calculated by a diagrammatic method that is called the “refined topological vertex”. We compute the extended A-model amplitudes for $SU(N)$ -geometries using the proposed vertex. If the refined topological vertex is valid, these computations should give rise to the Nekrasov’s partition functions of $\mathcal{N}=2$ $SU(N)$ gauge theories via the geometric engineering. In this article, we verify the proposal by confirming the equivalence between the refined A-model amplitude and the K-theoretic version of the Nekrasov’s partition function by explicit computation.

1 Introduction

The study of topological A-model strings on non-compact toric Calabi-Yau manifolds has been the important subject in the research of topological strings. Topological strings has provided insights into mathematics and nonperturbative dynamics of gauge and string theory.

On the one hand, it is in general very hard to calculate the topological string partition functions exactly. However in some cases, various dualities enable us to simplify the calculation of topological strings greatly and provide new perspectives [1]. For example, the A-model partition function on the resolved conifold is given by the partition function of Chern-Simons theory on S^3 . This is the geometric transition between the resolved conifold and the deformed conifold [2][3][4]. By generalizing this argument, an elegant technique for computing the A-model partition function on toric Calabi-Yau manifolds was formulated in [5]. The formalism is called the topological vertex.

The mechanism of the geometric engineering is one way to study supersymmetric gauge theories using string theory and topological string [6]. This approach tells us that we can calculate the F-terms of various $\mathcal{N}=2$ $SU(N)$ gauge theories by using topological A-model strings on certain toric Calabi-Yau manifolds. The partition function of the A-model on the toric Calabi-Yau agrees with the Nekrasov's partition function of $\mathcal{N}=2$ $SU(N)$ gauge theory [7][8][9][10][11][12][13]. Thus topological strings are useful tool to obtain insights into the nonperturbative dynamics of supersymmetric gauge theories.

The Nekrasov's partition function in a constant self-dual graviphoton background contains one parameter which is corresponding to the value of the background field. The parameter is nothing but the topological string coupling constant in A-model side. On the other hand, we can perform the instanton calculation in the more general background of non self-dual graviphoton configuration, and we get the K-theoretic version of the Nekrasov's partition function [7][14][15]. Then the Nekrasov's partition function has one more parameter in addition to the self-dual graviphoton background. Hence it is natural to expect that there exists a 2-parameter extension of the topological vertex which will recover the K-theoretic answer. Few attempts were made for defining the 2-parameter extension of topological strings and formulating the algorithmical techniques to calculate the extended partition function [11][16][17]. Recently a refined topological vertex was proposed in [18]. In this article, we compute the refined topological A-model string partition function for the $SU(N)$ geometries and check the equivalence of the refined partition function and the K-theoretic version of the Nekrasov's partition function.

This paper is organized as follows. In section 2, we review the geometric engineering, the topological vertex and their 2-parameter extension. The refined A-model partition function for $SU(N)$ geometries are calculated in section 3. Conclusions are found in section 4. In appendix A,

we give brief introduction to Young diagrams, Schur functions, and the useful formulae for Schur functions. In appendix B, a proof of a formula can be found.

2 Topological Strings and Instanton Counting

In this section, we will briefly review the idea of the geometric engineering, topological A-model strings, and the instanton counting.

2.1 Geometric Engineering and A-model

Type IIA string theory compactified on a Calabi-Yau 3-fold yields an effective theory in transverse 4-dimensions. Especially, enhanced gauge symmetries arise from singular Calabi-Yau compactification. Thus in the field theory limit, appropriate Calabi-Yau compactifications provide effective gauge theories in 4-dimensions. This is the basic idea of the geometric engineering [6].

Let us consider Type IIA compactified on a Calabi-Yau 3-fold M . The Kähler parameters of M are denoted by t_i . Then, the F-term of the effective theory is given by [19][20]

$$\sum_{g=0}^{\infty} \int d^4x d^4\theta W^{2g} F_g(t_i) = \int d^4x \left[\tau_{ij} F_{\mu\nu}^i F^{j\mu\nu} + \sum_{g=1}^{\infty} F_g(t_i) R_+^2 F_+^{2g-2} \right] \quad (2.1)$$

Here, W is $W_{\mu\nu} = F_{\mu\nu}^+ - R_{\mu\nu\rho\sigma} \theta \sigma^{\rho\sigma} \theta + \dots$, F_+ is the self-dual part of the graviphoton field strength, R_+ is the self-dual part of the Riemann tensor, and $F_{\mu\nu}^i$ is the $U(1)$ gauge field strength of the effective theory. Notice that the 4-dimensional $U(1)$ gauge couplings are given by

$$\tau_{ij} = \frac{\partial^2}{\partial t_i \partial t_j} F_0(t_i) \quad (2.2)$$

Hence the genus zero amplitudes of Type IIA strings $F_0(t_i)$ give rise to the effective gauge couplings. This is the Seiberg-Witten theory [21] in Type IIA string theory set-up. The higher genus amplitudes $F_g(t_i)$ correspond to the graviphoton corrections to the gauge theory. They play an important role in the Nekrasov's partition function that gives a closed expression for the Seiberg-Witten prepotential [7][14][15].

Furthermore, the amplitudes of Type IIA strings $F_g(t_i)$ are identical with the topological A-model string amplitudes $\mathcal{F}_g(t_i)$ of M which "count" the holomorphic maps from genus g Riemann surfaces to a Calabi-Yau M [19][20]. The information of the partition function was encoded in the Gromov-Witten invariants. The generating function of these amplitudes is called the topological A-model string partition function

$$Z = \exp(\mathcal{F}(g_s, t_i)) = \exp \left(\sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(t_i) \right) \quad (2.3)$$

Here g_s is the topological string couplings constant.

2.2 Gopakumar-Vafa Invariants

The target space perspective tells us that we can reformulate A-model as BPS state counting problem. Let us consider M-theory lift of Type IIA on a Calabi-Yau, i.e. M-theory compactified on a Calabi-Yau times a circle. This set-up gives rise to an effective field theory in the transverse 5-dimension $\mathbb{R}^{1,3} \times S^1$. The particles in the effective theory arise from M2 branes wrapping holomorphic curves of M . The mass and the charge (j_L, j_R) of the little group in 5-dimensions $SO(4) = SU(2)_L \times SU(2)_R$ characterise these BPS particles. The masses are given by $m_{(\Sigma, n)} = T_\Sigma + \frac{2\pi i n}{g_s}$. Here T_Σ is the Kähler parameter of the curve class Σ which M2 brane wraps, and n is the momentum along S^1 . Therefore the mass (and charge via BPS condition) is given by the curve class Σ and the momentum n . Integrating out these particles, we get the F-term of the effective theory [3]

$$\begin{aligned} \mathcal{F} &= \sum_{\Sigma \in H_2(M, \mathbb{Z})} \sum_{n \in \mathbb{Z}} \sum_{j_L, j_R} N_\Sigma^{(j_L, j_R)} \log \det_{(j_L, j_R)} (\Delta + m_{(\Sigma, n)}^2 + 2m_{(\Sigma, n)} \sigma_L F_+) \\ &= \sum_{\Sigma \in H_2(M, \mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L} N_\Sigma^{j_L} e^{-k T_\Sigma} \frac{\text{Tr}_{j_L} (-1)^{\sigma_L} e^{-2k g_s \sigma_L}}{k (2 \sinh(k g_s / 2))^2} \\ &= \sum_{\Sigma \in H_2(M, \mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L} N_\Sigma^{j_L} (-1)^{-2j_L} e^{-k T_\Sigma} \frac{\sum_{l=-j_L}^{j_L} q^{-2kl}}{k (q^{k/2} - q^{-k/2})^2} \end{aligned}$$

Notice that the graviphoton expectation value gives topological string coupling $F_+ = g_s$ and we introduce $q = e^{-g_s}$. Changing representation basis of $SU(2)_L$ so as to satisfy $\sum_{j_L} N_\Sigma^{j_L} (\sum_{l=-j_L}^{j_L} q^l) = \sum_{g=0}^{\infty} n_\Sigma^g (-1)^g (q^{1/2} - q^{-1/2})^{2g}$, we get the following expression of the A-model partition function

$$\mathcal{F} = \log Z = \sum_{\Sigma \in H_2(M, \mathbb{Z})} \sum_{k=1}^{\infty} \sum_{g=0}^{\infty} \frac{n_\Sigma^g}{k} (q^{k/2} - q^{-k/2})^{2g-2} e^{-T_\Sigma} \quad (2.4)$$

Integer valued invariants n_Σ^g which are defined as above are called "Gopakumar-Vafa invariants".

$N_\Sigma^{(j_L, j_R)}$ is the number of the wrapped M2-branes, and they are not invariant under the complex structure deformations of Calabi-Yau. Roughly speaking, this is the reason why the information encoded in the partition function is not the full degeneracies $N_\Sigma^{(j_L, j_R)}$ but $N_\Sigma^{j_L}$ which are summed over $SU(2)_R$ charges as

$$N_\Sigma^{j_L} = \sum_{j_R} (-1)^{-2j_R} (2j_R + 1) N_\Sigma^{(j_L, j_R)} \quad (2.5)$$

However $N_{\Sigma}^{(j_L, j_R)}$ themselves are invariants for non-compact Calabi-Yau since these Calabi-Yau 3-folds have no complex structure deformations [11]. Among them, local toric Calabi-Yau 3-folds are important ones. Hence we define an extended partition function that counts invariants $N_{\Sigma}^{(j_L, j_R)}$ as follows

$$\begin{aligned} \mathcal{F} &= \sum_{\Sigma \in H_2(M, \mathbb{Z})} \sum_{n \in \mathbb{Z}} \sum_{j_L, j_R} N_{\Sigma}^{(j_L, j_R)} \log \det_{(j_L, j_R)} (\Delta + m_{(\Sigma, n)}^2 + 2m_{(\Sigma, n)} \sigma_L (F_+ + F_-)) \\ &= \sum_{\Sigma \in H_2(M, \mathbb{Z})} \sum_{k=1}^{\infty} \sum_{j_L, j_R} N_{\Sigma}^{(j_L, j_R)} (-1)^{-2(j_L + j_R)} e^{-k T_{\Sigma}} \frac{\left(\sum_{l=-j_L}^{j_L} (tq)^{-2kl} \right) \left(\sum_{m=-j_R}^{j_R} \left(\frac{t}{q} \right)^{-2km} \right)}{k (t^{k/2} - t^{-k/2}) (q^{k/2} - q^{-k/2})} \end{aligned}$$

Here $q = e^{F_+}$ and $t = e^{F_-}$.

The question now arises; how to compute these partition functions for non-compact Calabi-Yau. In the case of toric Calabi-Yau 3-folds, the answer can be found in a diagrammatic methods named the topological vertex. Before we turn to the discussion of topological vertex, it will be useful to take a look at the instanton counting of $\mathcal{N}=2$ gauge theory. Hence in the next section, we discuss the Nekrasov's partition function of $\mathcal{N}=2$ gauge theory. We will come back to the discussion of the topological vertex later.

2.3 Instanton Counting of $\mathcal{N}=2$ Gauge Theories

Instanton calculation of $\mathcal{N}=2$ gauge theories in 4- and 5-dimensions has been developed by Nekrasov [7]. He found that the instanton coefficients of the Seiberg-Witten prepotential are summed up to a closed form, and he provided the combinatorial expression of this generating function. We call it the Nekrasov's partition function. His conjectural observation was mathematically verified by Nekrasov-Okounkov [23], Nakajima-Yoshioka [24], and Braverman [25].

Take an $\mathcal{N}=2$ $SU(N)$ supersymmetric pure Yang-Mills theory for example. Multi-instanton calculation involves an integral over the ADHM moduli space. It is in general very hard to carry out the calculation. However we can formulate the multi-instanton calculation of $\mathcal{N}=2$ $SU(N)$ supersymmetric gauge theory as integrals of equivariant closed forms. Let us consider the following partition function of $\mathcal{N}=2$ $SU(N)$ supersymmetric pure Yang-Mills theory

$$Z^{inst.}(\vec{a}, \Lambda) = \sum_{k=1}^{\infty} \Lambda^{2Nk} Z^k(\vec{a})$$

Here \vec{a} is the Coulomb moduli, Λ is the dynamical scale, and $Z^k(\vec{a})$ is a k -instanton contribution. By deforming the theory by torus action on the moduli space, we can give the partition function

as an integral of equivariant differential

$$Z^k = \int_{\mathcal{M}(N,k)} \mathcal{D}\mu e^{-Q\Psi} \quad (2.6)$$

where $\mathcal{M}(N, k)$ is the ADHM moduli space of k -instantons and Q is the BRST operator. It is known that the BRST operator is an equivariant differential for torus action $T = U(k) \times U(1)^{N-1} \times U(1)^2$ on the moduli space. Here $U(1)^2$ is the rotation groups of complex plane $\mathbb{R}^4 = \mathbb{C}^2$ and their weights provide deformation parameters ϵ_i . Then we can apply the localization formula

$$Z^k = \sum_{p_0} \frac{1}{\sqrt{\det \mathcal{L}_{p_0}}} \quad (2.7)$$

Here p_0 are isolated fixed points of the torus action and \mathcal{L}_{p_0} is the Lie derivative acting on the tangent moduli space $T\mathcal{M}(N, k)$. It is known that the fixed points of T -action are uniquely specified by N Young diagrams (μ_1, \dots, μ_N) . Then we have to know the weights $\det \mathcal{L}_{p_0}$ of T -action on the tangent moduli space $T\mathcal{M}(N, k)$ for the purpose of multi-instanton calculus. The weights were calculated in [26][7][14][15] and the explicit expression is given by

$$\begin{aligned} Z^{inst.}(\epsilon_1, \epsilon_2, \vec{a}, \Lambda) &= \sum_{\vec{\mu}} \Lambda^{2N|\vec{\mu}|} \prod_{a,b=1}^N \prod_{s \in \mu_a} \frac{1}{a_b - a_a - \epsilon_1 l_{\mu_b}(s) + \epsilon_2(a_{\mu_a}(s) + 1)} \\ &\times \prod_{t \in \mu_b} \frac{1}{a_b - a_a + \epsilon_1(l_{\mu_a}(s) + 1) - \epsilon_2 a_{\mu_b}(s)} \end{aligned} \quad (2.8)$$

Nekrasov claimed that the partition function (2.8) leads to the Seiberg-Witten prepotential after eliminating the deformation parameter ϵ as follows

$$\epsilon_1 \epsilon_2 \log Z^{inst.}(\epsilon_1, \epsilon_2, \vec{a}, \Lambda) = \mathcal{F}_{SW}^{inst.}(\vec{a}, \Lambda) + O(\epsilon_1, \epsilon_2) \quad (2.9)$$

This conjecture was proved by using the thermodynamical limit of the random partition [23], the blow-up equation [24], and [25].

We can lift it to the 5-dimensional gauge theory result

$$Z_{5D}^{inst.}(t, q, \vec{a}, \Lambda, \beta) = \sum_{\vec{\mu}} (\beta \Lambda)^{2N|\vec{\mu}|} \prod_{a,b=1}^N \prod_{s \in \mu_a} \frac{1}{1 - Q_{ba} t^{l_{\mu_b}(s)} q^{a_{\mu_a}(s)+1}} \prod_{t \in \mu_b} \frac{1}{1 - Q_{ba} t^{-l_{\mu_a}(s)-1} q^{-a_{\mu_b}(s)}} \quad (2.10)$$

Here β is the radius of the compact fifth dimension S^1 . Let us choose the deformation parameter ϵ as $\hbar = \epsilon_1 = -\epsilon_2$. In [10][9][11] the partition function was reproduced from the string calculation via the geometric engineering

$$Z_{5D}^{Nek.SU(N)}(\hbar, -\hbar, \vec{a}, \Lambda, \beta) = Z^{A-model, SU(N)}(\hbar = g_s, t_i) \quad (2.11)$$

and they verified the interpretation in [7] that \hbar expansion is nothing but the genus expansion of the string partition function. Notice that the Coulomb moduli \vec{a} and the dynamical scale Λ are engineered from the Kähler parameters of the Calabi-Yau. We review the results (2.11) for $SU(2)$ theory later.

Thus it is natural to expect that there is a refinement of string theory to engineer Nekrasov' partition function for the general case $\epsilon_1 \neq -\epsilon_2$. In this paper we calculate the K-theoretic partition function (2.10) via the refined topological vertex and show that the refined A-model of [18] reproduces the correct results.

2.4 Topological Vertex and its refinement

It is known that we can compute the topological A-model string amplitudes for toric Calabi-Yau 3-folds by using the topological vertex [5]. The topological vertex is the Feynman-rules like technique which arises from the geometric transition between A-model and Chern-Simons gauge theory. The Feynman diagrams, the vertices of diagrams, the momentum, and the propagators are corresponding to the toric web-diagrams, the tri-valent vertices $C_{\mu_1 \mu_2 \mu_3}$, Young diagrams μ , and the weights $(-1)^{(n+1)|\mu|} e^{-T|\mu|} q^{-\frac{n\kappa_\mu}{2}}$, respectively. Here, T is the Kähler parameter for the 2-cycle corresponding to the line of the web-diagram, μ is the Young diagram which propagates along the line. The framing number n is determined by the toric diagram. The vertex is expressed using the Schur functions

$$C_{\lambda\mu\nu}(q) = q^{\kappa_\mu/2} s_{\nu^t}(q^{-\rho}) \sum_{\eta} s_{\lambda^t/\eta}(q^{-\nu-\rho}) s_{\mu/\eta}(q^{-\nu^t-\rho}) \quad (2.12)$$

The vertices in Fig.1 are glued as

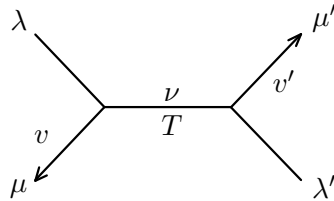


Figure 1: The toric diagram obtained by gluing the vertices $C_{\lambda\mu\nu}$ and $C_{\lambda'\mu'\nu^t}$

$$\sum_{\nu} C_{\lambda\mu\nu}(q) (-1)^{(n+1)|\nu|} q^{-n\kappa_\nu/2} e^{-T|\nu|} C_{\lambda'\mu'\nu^t}(q) \quad (2.13)$$

where the framing number n is given by $n = v' \wedge v = v'_1 v_2 - v_1 v'_2$.

The local Hirzebruch surface $\mathbb{F}_0 = C(\mathbb{P}^1 \times \mathbb{P}^1)$ is a good example to illustrate the topological vertex calculation. This toric Calabi-Yau 3-fold is the typical $SU(2)$ geometry that engineers $SU(2)$

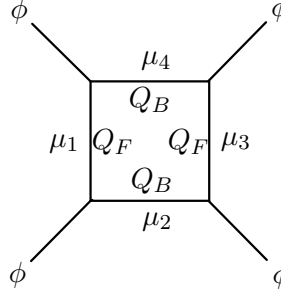


Figure 2: The local Hirzebruch surface which is a line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$

pure super Yang-Mills theory. The toric diagram is given by Fig.2, and we can easily check that the framing numbers associated to the four internal lines are all 1. Applying the topological vertex to Fig.2, we get the following partition function

$$\begin{aligned}
Z^{F_0}(Q_F, Q_B) &= \sum_{\mu_1, \mu_2, \mu_3, \mu_4} Q_F^{|\mu_1|+|\mu_3|} Q_B^{|\mu_2|+|\mu_4|} q^{-\kappa_{\mu_1}/2 + \kappa_{\mu_2}/2 - \kappa_{\mu_3}/2 - \kappa_{\mu_4}/2} \\
&\quad \times C_{\phi \mu_1 \mu_4^t} C_{\phi \mu_2^t \mu_1^t} C_{\mu_2 \phi \mu_3} C_{\phi \mu_4 \mu_3^t} \\
&= \sum_{\mu_2, \mu_4} Q_B^{|\mu_2|+|\mu_4|} q^{+\kappa_{\mu_2}/2 - \kappa_{\mu_4}/2} K_{\mu_4 \mu_2}(Q_F) K_{\mu_2^t \mu_4^t}(Q_F)
\end{aligned}$$

$K_{\mu\nu}$ is defined as follows

$$\begin{aligned}
K_{\mu\nu} &= \sum_{\lambda} Q_F^{|\lambda|} q^{-\kappa_{\lambda}/2} C_{\phi \lambda \mu^t} C_{\nu^t \lambda^t \phi} \\
&= s_{\mu^t}(q^{-\rho}) s_{\nu}(q^{-\rho}) \sum_{\lambda} Q_F^{|\lambda|} s_{\lambda}(q^{-\mu-\rho}) s_{\lambda}(q^{-\nu^t-\rho}) \\
&= q^{\|\mu\|^2/2 + \|\nu^t\|^2/2} \tilde{Z}_{\mu^t}(q) \tilde{Z}_{\nu}(q) \prod_{i,j=1}^{\infty} \frac{1}{1 - Q_F q^{-\mu_i - \nu_j^t + i + j - 1}}
\end{aligned}$$

where we use the relation $s_{\mu}(q^{-\rho}) = q^{\|\mu\|^2/2} \prod_{s \in \mu} (1 - q^{h_{\mu}(s)})^{-1} = q^{\|\mu\|^2/2} \tilde{Z}_{\mu}(q)$ and formula (A.17).

Let us separate out the perturbative contributions as

$$Z^{\mathbb{F}_0}(Q_B, Q_F) = Z_{pert.}^{\mathbb{F}_0}(Q_F) Z_{inst.}^{\mathbb{F}_0}(Q_B, Q_F) \quad (2.14)$$

$$Z_{pert.}^{\mathbb{F}_0}(Q_F) \equiv K_{\phi\phi}(Q_F)^2 = \left[\prod_{i,j=1}^{\infty} \frac{1}{1 - Q_F q^{i+j-1}} \right]^2 \quad (2.15)$$

Then, we get the A-model partition function corresponding to the nonperturbative part of the Nekrasov's partition function

$$Z_{inst.}^{\mathbb{F}_0} = \sum_{\mu, \nu} Q_B^{|\mu|+|\nu|} q^{\|\mu\|^2 + \|\nu^t\|^2} \tilde{Z}_{\mu}(q) \tilde{Z}_{\mu^t}(q) \tilde{Z}_{\nu}(q) \tilde{Z}_{\nu^t}(q) \left[\prod_{i,j=1}^{\infty} \frac{1 - Q_F q^{i+j-1}}{1 - Q_F q^{-\mu_i - \nu_j^t + i + j - 1}} \right]^2 \quad (2.16)$$

In fact, applying the formula (3.19) for the special case we can show that the above result is identical with the Nekrasov's partition function of the $SU(2)$ Yang-Mills theory (2.10) for $t = q$. The identifications of parameters are given by

$$Q_B = (\beta\Lambda)^4, \quad Q_F = e^{2\beta a} \quad (2.17)$$

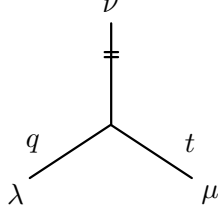


Figure 3: The refined topological vertex $C_{\lambda\mu\nu}(t, q)$

Recently, the topological vertex formalism for the refined partition functions has been proposed in [18] via melting crystal picture of the topological vertex. We call it the refined topological vertex. It was claimed that the refined topological vertex is constructed so as to engineer the K-theoretic version of the Nekrasov's partition function. We verify this claim in the next section. The proposal of [18] is as follows: the refined vertex corresponding to Fig.3 is given by

$$C_{\lambda\mu\nu}(t, q) = \left(\frac{q}{t}\right)^{\frac{\|\mu\|^2 + \|\nu\|^2}{2}} t^{\frac{\kappa_\mu}{2}} P_{\nu^t}(t^{-\rho}; q, t) \sum_{\eta} \left(\frac{q}{t}\right)^{\frac{|\eta| + |\lambda| - |\mu|}{2}} s_{\lambda^t/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t} q^{-\rho}) \quad (2.18)$$

and we glue the "t-edge" and the "q-edge" with weight

$$f_{\mu}(t, q) = (-1)^{|\mu|} t^{n(\mu)} q^{-n(\mu^t)} \quad (2.19)$$

The purpose of this article is to confirm that the refined vertex for $SU(N)$ geometry engineers the K-theoretic version of the Nekrasov's partition function. The refined partition functions for the $SU(2)$ and $SU(3)$ geometries and their blow-up were computed in [18]. Hence in the next section, we generalize their discussion to the general $SU(N)$ geometries and their blow-up. As the result, we propose that we should modify the framing factors in order to engineer the Nekrasov's results.

3 Refined A-model Amplitudes and Nekrasov's partition functions

In this section, we compute the refined partition function for $SU(N)$ geometry via refined topological vertex. The important point of the result in this section is that these refined partition

functions are the same as the K-theoretic version of the Nekrasov's partition functions under the little modification of the framing factor. This result verifies the proposal of the refined topological vertex.

3.1 $\mathcal{N} = 2$ $SU(N)$ Super Yang-Mills

3.1.1 A-model Partition Function

The toric diagram of $SU(N)$ geometry which give rise to the $\mathcal{N} = 2$ $SU(N)$ super Yang-Mills are shown in Fig.4(a). The parallel edges corresponding to the base \mathbb{P}^1 are the preferred directions of [18]. For fixed N , there are $N + 1$ inequivalent geometries ($m = 0 \cdots N$) which give $SU(N)$ super Yang-Mills. The number m is the Chern-Simons coefficient of the 5-dimensional theory in the gauge theory side.

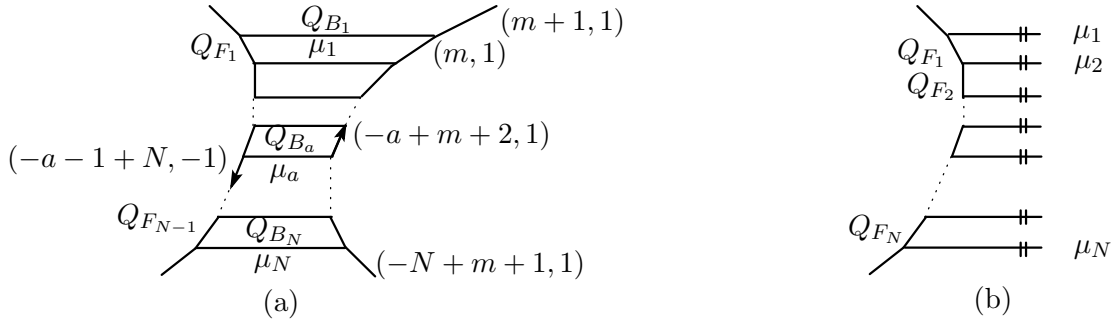


Figure 4: (a)The toric diagram of $SU(N)$ geometry (b)The building block of $SU(N)$ geometry, and refined vertex on this geometry implies $K_{\mu_1 \cdots \mu_N}(Q_{F,1}, \cdots, Q_{F,N-1})$

Let us start with the computation of the subdiagram Fig.4(b). For the reason which we discuss later, we modify slightly the framing factor proposed in [18] as follows

$$f_\mu(t, q) = (-1)^{|\mu|} t^{\frac{\|\mu^t\|^2}{2}} q^{-\frac{\|\mu\|^2}{2}} = (-1)^{|\mu|} \left(\frac{t}{q}\right)^{\frac{\|\mu^t\|^2}{2}} q^{-\frac{\kappa_\mu}{2}} \quad (3.1)$$

Using the refined vertex, we can express the subdiagram as

$$\begin{aligned} K_{\mu_1 \cdots \mu_N}(Q_{F,1}, \cdots, Q_{F,N-1}) &= \sum_{\lambda_1 \cdots \lambda_{N-1}} \prod_{a=1}^N (-Q_{F,a})^{|\lambda_a|} f_{\lambda_a}(t, q) C_{\lambda_{a-1} \lambda_a \mu_a}(t, q) \\ &= \sum_{\lambda_1 \cdots \lambda_{N-1}} \prod_{a=1}^N \sum_{\eta_1 \cdots \eta_N} Q_{F,a}^{|\lambda_a|} \left(\frac{t}{q}\right)^{\frac{\|\lambda_a^t\|^2}{2}} q^{-\frac{\kappa_{\lambda_a}}{2}} \left(\frac{q}{t}\right)^{\frac{\|\lambda_a\|^2 + \|\mu_a\|^2}{2}} \\ &\quad \times t^{\frac{\kappa_{\lambda_a}}{2}} P_{\mu_a^t}(t^{-\rho}; q, t) \left(\frac{q}{t}\right)^{\frac{|\eta_a| + |\lambda_{a-1}| - |\lambda_a|}{2}} \\ &\quad \times s_{\lambda_{a-1}/\eta_a}(t^{-\rho} q^{-\mu_a}) s_{\lambda_a/\eta_a}(t^{-\mu_a^t} q^{-\rho}) \end{aligned} \quad (3.2)$$

Notice $\|\mu\|^2 - \|\mu^t\|^2 = \kappa_\mu$ and $\lambda_0 = \lambda_N = \phi$. Simplifying the summation, we get

$$K_{\mu_1 \dots \mu_N} (Q_{F,1}, \dots, Q_{F,N-1}) = \prod_{a=1}^N \left[q^{\frac{\|\mu_a\|^2}{2}} \tilde{Z}_{\mu_a}(t, q) \right] \\ \times \sum_{\lambda_1 \dots \lambda_{N-1}} \sum_{\eta_1 \dots \eta_N} \prod_{a=1}^N Q_{F,a}^{|\lambda_a|} \left(\frac{q}{t} \right)^{\frac{|\eta_a|}{2}} s_{\lambda_{a-1}/\eta_a}(t^{-\rho} q^{-\mu_a}) s_{\lambda_a/\eta_a}(t^{-\mu_a^t} q^{-\rho})$$

The sum involved in the above subdiagram becomes

$$\sum_{\lambda_1 \dots \lambda_{N-1}} \sum_{\eta_1 \dots \eta_N} \prod_{a=1}^N \left(\sqrt{\frac{q}{t}} Q_{F,a} \right)^{|\lambda_a|} s_{\lambda_{a-1}/\eta_a}(t^{-\rho} q^{-\mu_a - \frac{1}{2}}) s_{\lambda_a/\eta_a}(t^{-\mu_a^t + \frac{1}{2}} q^{-\rho}) \\ = \sum_{\lambda_1 \dots \lambda_{N-1}} \sum_{\rho_1 \dots \rho_{N-1}} \prod_{a=1}^{N-1} \left(\sqrt{\frac{q}{t}} Q_{F,a} \right)^{|\lambda_a|} s_{\lambda_a/\rho_{a-1}}(t^{-\mu_a^t + \frac{1}{2}} q^{-\rho}) s_{\lambda_a/\rho_a}(t^{-\rho} q^{-\mu_{a+1} - \frac{1}{2}})$$

Notice that $\rho_0 = \rho_{N-1} = \phi$. We can take the summation over Young diagrams by Lemma 3.1 of [12], or using the vertex on a strip [27] as we will discuss in the next subsection. Then we get

$$K_{\mu_1 \dots \mu_N} (Q_{F,a}) = \prod_{a=1}^N \left[q^{\frac{\|\mu_a\|^2}{2}} \tilde{Z}_{\mu_a}(t, q) \right] \\ \times \prod_{1 \leq a < b \leq N} \prod_{i,j=1}^{\infty} \frac{1}{1 - Q_{ab} t^{-\mu_{ai}^t + j} q^{-\mu_{bj} + i - 1}} \quad (3.3)$$

where $Q_{ab} \equiv \prod_{l=a}^{b-1} \left(\sqrt{\frac{q}{t}} Q_{F,l} \right) \equiv \prod_{l=a}^{b-1} \tilde{Q}_{F,l}$.

Let us glue these subdiagrams. The framing factors are given by $n_a = (a - m - 2, -1) \wedge (a - N + 1, 1) = -(N + m - 2a + 1)$ as Fig.4. Then, the A-model amplitude is

$$Z^{A\text{-model}, SU(N)} (Q_B, Q_{F,a}) = \sum_{\mu_1 \dots \mu_N} \prod_{a=1}^N \left[Q_{B,a}^{|\mu_a|} f_{\mu_a}(t, q)^{n_a} \right] K_{\mu_1 \dots \mu_N} (Q_{F,a}, t, q) K_{\mu_N^t \dots \mu_1^t} (Q_{F,a}, q, t) \\ = Z_{pert.}^{A\text{-model}, SU(N)} (Q_{F,a}) Z_{inst.}^{A\text{-model}, SU(N)} (Q_B, Q_{F,a}) \quad (3.4)$$

The perturbative part of the partition function is given by [7]

$$Z_{pert.}^{A\text{-model}, SU(N)} (Q_{F,a}) \equiv K_{\phi \dots \phi} (Q_{F,a})^2$$

By substituting (3.3) into (3.4), we obtain

$$Z_{inst.}^{A\text{-model}, SU(N)} (Q_B, Q_{F,a}) = \sum_{\mu_1 \dots \mu_N} \prod_{a=1}^N \left[Q_{B,a}^{|\mu_a|} f_{\mu_a}(t, q)^{n_a} q^{\frac{\|\mu_a\|^2}{2}} t^{\frac{\|\mu_a^t\|^2}{2}} \tilde{Z}_{\mu_a}(t, q) \tilde{Z}_{\mu_a^t}(q, t) \right] \\ \times \prod_{1 \leq a < b \leq N} \prod_{i,j=1}^{\infty} \frac{1 - Q_{ab} t^j q^{i-1}}{1 - Q_{ab} t^{-\mu_{ai}^t + j} q^{-\mu_{bj} + i - 1}} \frac{1 - Q_{ab} t^{j-1} q^i}{1 - Q_{ab} t^{-\mu_{ai}^t + j - 1} q^{-\mu_{bj} + i}} \quad (3.5)$$

As we show in the next subsection, the partition function is identical with that of Nekrasov.

3.1.2 Identification with Nekrasov's Partition Functions

In this subsection, we show that the refined A-model amplitude agrees with the K-theoretic version of the Nekrasov's partition function:

$$Z_{inst.}^{A-model, SU(N)}(Q_B, Q_{F,a}) = Z_{inst.}^{Nek, SU(N)}(\hat{Q}, Q_{ab}) \quad (3.6)$$

Recall that the K-theoretic version of the Nekrasov's partition functions with a Chern-Simons term is given by [28][29]

$$Z_{inst.}^{Nek, SU(N), m}(\hat{Q}, Q_{ab}) = \sum_{\vec{\mu}} \frac{\hat{Q}^{|\vec{\mu}|}}{\prod_{a,b} N_{ab}^{\vec{\mu}}(t, q, Q_{ab})} \left(\frac{q}{t}\right)^{\frac{N}{2}|\vec{\mu}|} \prod_{a=1}^N e_a^{m|\mu_a|} t^{-m\frac{\|\mu_a^t\|^2}{2}} q^{m\frac{\|\mu_a\|^2}{2}} \quad (3.7)$$

Note that $Q_{ab} = e_a e_b^{-1}$.

First, let us rewrite the character part $\prod N_{ab}^{\vec{\mu}}$. The identity

$$\sum_{i,j=1}^{\infty} q^{\mu_i - j + 1} t^{\nu_j - i} = \sum_{i,j=1}^{\infty} q^{-\nu_j^t + i} t^{-\mu_i^t + j - 1} \quad (3.8)$$

follows from $(t-1) \sum_{i=1}^{\infty} q^{\mu_i} t^{-i} = (q^{-1}-1) \sum_{i=1}^{\infty} t^{-\mu_i^t} q^i$ for $t, q \neq 1$ [16]. It is easy to prove the following formula using (3.8) (take the logarithm of the equation(3.9))

$$\prod_{i,j=1}^{\infty} \left(1 - Q t^{-\mu_j^t + i} q^{-\nu_i + j - 1}\right) = \prod_{i,j=1}^{\infty} \left(1 - Q q^{\mu_i - j} t^{\nu_j^t - i + 1}\right) \quad (3.9)$$

The character part of the Nekrasov's partition function is given by

$$\begin{aligned} \frac{1}{N_{12}^{\vec{\mu}}(t, q, Q)} &\equiv \prod_{(i,j) \in \mu} \frac{1}{1 - Q t^{\nu_j^t - i} q^{\mu_i - j + 1}} \prod_{(i,j) \in \nu} \frac{1}{1 - Q t^{-\mu_j^t + i - 1} q^{-\nu_i + j}} \\ &= \prod_{i,j=1}^{\infty} \frac{1 - Q t^{j-1} q^i}{1 - Q t^{-\mu_j^t + i - 1} q^{-\nu_i + j}} \end{aligned} \quad (3.10)$$

where $\mu_1 = \mu, \mu_2 = \nu, Q_{12} = Q$. By using (3.9), we have

$$\begin{aligned} \prod_{i,j=1}^{\infty} \frac{1 - Q t^j q^{i-1}}{1 - Q t^{-\mu_j^t + i} q^{-\nu_i + j - 1}} &= \prod_{i,j=1}^{\infty} \frac{1 - Q t^j q^{i-1}}{1 - Q q^{\mu_i - j} t^{\nu_j^t - i + 1}} \\ &= \prod_{(i,j) \in \nu} \frac{1}{1 - Q t^{-\mu_j^t + i} q^{-\nu_i + j - 1}} \prod_{(i,j) \in \mu} \frac{1}{1 - Q t^{\nu_j^t - i + 1} q^{\mu_i - j}} \\ &= (-Q)^{-|\mu| - |\nu|} t^{\sum_{(i,j) \in \nu} (\mu_j^t - i) - \sum_{(i,j) \in \mu} (\nu_j^t - i + 1)} q^{\sum_{(i,j) \in \nu} (\nu_i - j + 1) - \sum_{(i,j) \in \mu} (\mu_i - j)} \\ &\quad \times \frac{1}{N_{21}^{\vec{\mu}}(t, q, Q^{-1})} \end{aligned} \quad (3.11)$$

The factors appear in the above equation become

$$\sum_{(i,j) \in \nu} \mu_j^t = \sum_{j=1}^{\nu_1} \sum_{i=1}^{\nu_j^t} \mu_j^t = \sum_{j=1}^{\min(\mu_1, \nu_1)} \mu_j^t \nu_j^t = \sum_{(i,j) \in \mu} \nu_j^t$$

$$\sum_{(i,j) \in \mu} (\mu_i - j) = \sum_{i=1}^{d(\mu)} [(\mu_i - 1) + \dots + (\mu_i - \mu_i)] = \frac{\|\mu\|^2}{2} - \frac{|\mu|}{2}$$

Hence we obtain

$$\frac{1}{N_{12}^{\vec{\mu}}(t, q, Q) N_{21}^{\vec{\mu}}(t, q, Q^{-1})} = (-Q)^{|\mu|+|\nu|} \left(\frac{q}{t}\right)^{-\frac{|\mu|}{2} - \frac{|\nu|}{2} + \frac{\|\mu^t\|^2}{2} - \frac{\|\nu^t\|^2}{2}} q^{\frac{\kappa_\mu}{2} - \frac{\kappa_\nu}{2}}$$

$$\times \prod_{i,j=1}^{\infty} \frac{1 - Q_{12} t^{i-1} q^j}{1 - Q_{12} t^{-\mu_j^t + i - 1} q^{-\nu_i + j}} \frac{1 - Q_{12} t^i q^{j-1}}{1 - Q_{12} t^{-\mu_j^t + i} q^{-\nu_i + j - 1}} \quad (3.12)$$

It is easy to show

$$\frac{1}{N_{aa}^{\vec{\mu}}(t, q, Q_{aa} = 1)} = (-1)^{|\mu_a|} \left(\frac{t}{q}\right)^{\frac{|\mu_a|}{2}} t^{\frac{\|\mu_a^t\|^2}{2}} q^{\frac{\|\mu_a\|^2}{2}} \tilde{Z}_{\mu_a}(t, q) \tilde{Z}_{\mu_a^t}(q, t) \quad (3.13)$$

By combining above identities, we can rewrite the Nekrasov's partition function as follows

$$\sum_{\vec{\mu}} \frac{\hat{Q}^{|\vec{\mu}|}}{\prod_{a < b} N_{ab}^{\vec{\mu}}(t, q, Q_{ab})} \left(\frac{q}{t}\right)^{\frac{N}{2}|\vec{\mu}|} \prod_{a=1}^N e_a^{m|\mu_a|} t^{-m\frac{\|\mu_a^t\|^2}{2}} q^{m\frac{\|\mu_a\|^2}{2}}$$

$$= \sum_{\vec{\mu}} (-1)^{N|\vec{\mu}|} \hat{Q}^{|\vec{\mu}|} \prod_{a < b} [Q_{ab}^{|\mu_a|+|\mu_b|}] \prod_{a=1}^N [e_a^{m|\mu_a|} \left(\frac{q}{t}\right)^{(N+m-2a+1)\frac{\|\mu_a^t\|^2}{2}} q^{(N+m-2a+1)\frac{\kappa_{\mu_a}}{2}}$$

$$\times t^{\frac{\|\mu_a^t\|^2}{2}} q^{\frac{\|\mu_a\|^2}{2}} \tilde{Z}_{\mu_a}(t, q) \tilde{Z}_{\mu_a^t}(q, t)]$$

$$\times \prod_{a < b} \prod_{i,j=1}^{\infty} \frac{1 - Q_{ab} t^{i-1} q^j}{1 - Q_{ab} t^{-\mu_{aj}^t + i - 1} q^{-\mu_{bi} + j}} \frac{1 - Q_{ab} t^i q^{j-1}}{1 - Q_{ab} t^{-\mu_{aj}^t + i} q^{-\mu_{bi} + j - 1}} \quad (3.14)$$

Next, let us rewrite the remainder $\hat{Q}^{|\vec{\mu}|} \prod_{a < b} Q_{ab}^{|\mu_a|+|\mu_b|} \prod_{a=1}^N e_a^{m|\mu_a|}$. We shall rewrite it in terms of the Kähler parameters of the base and the fiber \mathbb{P}^1 's by showing the following identity

$$C Q_B^{|\vec{\mu}|} \prod_{a < b} Q_{ab}^{|\mu_a|+|\mu_b|} \prod_{a=1}^N e_a^{m|\mu_a|} = \prod_{a=1}^N Q_{B,a}^{|\mu_a|} \quad (3.15)$$

We prove this identity in the case of $N = \text{odd}$ and $m = \text{even}$ for example. It is easy to generalize this proof. First we use the results of [9], that is, $Q_{B,a}$ are given by the base and the fiber Kähler parameters and they satisfy

$$\prod_{i=1}^N Q_{B,a}^{|\mu_a|} = Q_B^{|\vec{\mu}|} \prod_{a=1}^{\lfloor \frac{N+m-1}{2} \rfloor} \tilde{Q}_{F,a}^{(N+m-2a)(|\mu_1|+\dots+|\mu_a|)} \prod_{a=\lfloor \frac{N+m}{2} \rfloor}^{N-1} \tilde{Q}_{F,a}^{-(N+m-2a)(|\mu_{a+1}|+\dots+|\mu_N|)} \quad (3.16)$$

Here we modify $Q_{F,a}$ to $\tilde{Q}_{F,a}$ in the case of the refined partition function.

On the one hand we can obtain the following identity after some algebra

$$\begin{aligned}
\prod_{a < b} Q_{ab}^{|\mu_a| + |\mu_b|} &= \left(\prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^a \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{N-a} \right)^{|\vec{\mu}|} \left(\prod_{a=1}^{\frac{N-1}{2}} \left(\prod_{b=a}^{\frac{N-1}{2}} \tilde{Q}_{F,b}^{(N-2b)} \right)^{|\mu_a|} \prod_{a=\frac{N+1}{2}+1}^{N-1} \left(\prod_{b=\frac{N+1}{2}}^{a-1} \tilde{Q}_{F,b}^{(2b-N)} \right)^{|\mu_a|} \right) \\
&= \left(\prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^a \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{N-a} \right)^{|\vec{\mu}|} \prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^{(N-2a)(|\mu_1| + \dots + |\mu_a|)} \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{(2a-N)(|\mu_{a+1}| + \dots + |\mu_N|)}
\end{aligned} \tag{3.17}$$

Here we use $Q_{ab} = \prod_{l=a}^{b-1} \tilde{Q}_{F,l}$. Using $\tilde{Q}_{F,a} = e_a e_{a+1}^{-1}$, we can also show

$$\begin{aligned}
&\prod_{a=1}^{\frac{N-1}{2}} \tilde{Q}_{F,a}^{m(|\mu_1| + \dots + |\mu_a|)} \prod_{a=\frac{N+1}{2}}^{N-1} \tilde{Q}_{F,a}^{-m(|\mu_{a+1}| + \dots + |\mu_N|)} \\
&\times \prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(N+m-2a)(|\mu_1| + \dots + |\mu_a|)} \prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(-2a+N+m)(|\mu_{a+1}| + \dots + |\mu_N|)} \\
&= \left(\prod_{a=1}^{\frac{N-1}{2}} \prod_{b=a}^{\frac{N-1}{2}} \tilde{Q}_{F,b}^{|\mu_a|} \prod_{a=\frac{N+1}{2}+1}^{N-1} \prod_{b=\frac{N+1}{2}}^{a-1} \tilde{Q}_{F,b}^{-|\mu_a|} \right)^m \left(\prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(N+m-2a)} \right)^{(|\mu_1| + \dots + |\mu_N|)} \\
&= \left(\left(e_{\frac{N+1}{2}} \right)^{-m} \prod_{a=\frac{N+1}{2}}^{\frac{N+m-1}{2}} \tilde{Q}_{F,a}^{(N+m-2a)} \right)^{|\vec{\mu}|} \prod_{a=1}^{N-1} e_a^{m|\mu_a|}
\end{aligned} \tag{3.18}$$

(3.16)(3.17)(3.18) provide the identity (3.15) immediately.

Finally, (3.14) (3.15) imply the following equality

$$Z_{inst.}^{A-model, SU(N)}(Q_B, Q_{F,a}) = Z_{inst.}^{Nek, SU(N)}(\hat{Q}, Q_{ab}) \tag{3.19}$$

where $\hat{Q} = (-1)^N C(e_a, t, q, m) Q_B$.

Note that if we use the framing factor without our modification, we get the additional factor

$$\prod_{a=1}^N \left(\frac{t}{q} \right)^{-\frac{n_a |\mu_a|}{2}} \tag{3.20}$$

in the summation of the refined patirion function. They cannot be absorbed into the \hat{Q} and break the equivalence (3.19). Hence we need the modification (3.1).

3.2 Adding Matters and Strip Geometries

By blowing up the $SU(N)$ geometries, we can add matters to the Nekrasov's instanton calculation via the geometric engineering. The Kähler parameters of the blown up \mathbb{P}^1 's give rise to the mass parameters of the matters. These geometries is obtained by gluing strip geometries. A strip geometry is a toric Calabi-Yau that contains a chain of \mathbb{P}^1 's. Each \mathbb{P}^1 locally forms a $(-1, -1)$ curve $O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1$ or $(-2, 0)$ curve $O(-2) \oplus O(0) \rightarrow \mathbb{P}^1$ as Fig.5. Following [27], we take

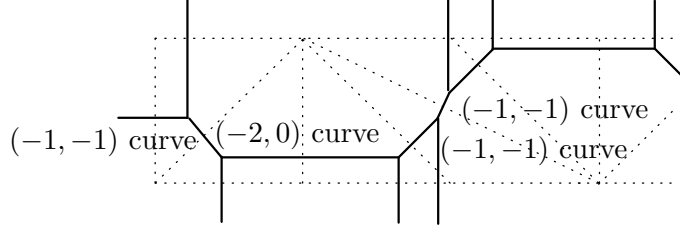


Figure 5: A toric diagram of a strip geometry which is obtained from triangulation of a strip toric data

the chain of $(-1, -1)$ curves Fig.6 for example. Gluing these strip geometries, we get the toric Calabi-Yau that engineers $\mathcal{N}=2$ $SU(N)$ gauge theory with $N_f = 2N$ [27]. The refined vertex on the strip geometry Fig.6(a) yields

$$\begin{aligned}
K_{\beta_1 \beta_2 \dots}^{\alpha_1 \alpha_2 \dots} &= \sum_{\{\mu_a\}, \{\nu_a\}} (-Q_{M,1})^{|\mu_1|} (-Q_{F,2})^{|\nu_2|} (-Q_{M,2})^{|\mu_2|} \dots \\
&\quad \times C_{\nu_1^t \mu_1 \alpha_1} C_{\nu_2^t \mu_1 \beta_1} C_{\nu_2^t \mu_2 \alpha_2} C_{\nu_3 \mu_2^t \beta_2} \times \dots \\
&= \prod_a \left[q^{\frac{\|\alpha_a\|^2}{2}} t^{\frac{\|\beta_a\|^2}{2}} \tilde{Z}_{\alpha_a}(t, q) \tilde{Z}_{\beta_a}(q, t) \right] \\
&\quad \times \sum_{\substack{\{\mu_a\}, \{\nu_a\} \\ \{\rho_a\}, \{\sigma_a\}}} \prod_a (-Q_{M,a})^{|\mu_a|} (-Q_{F,a})^{|\nu_a|} \left(\frac{q}{t}\right)^{\frac{\|\mu_a\|^2}{2}} t^{\frac{\kappa_{\mu_a}}{2}} \left(\frac{t}{q}\right)^{\frac{\|\mu_a^t\|^2}{2}} q^{-\frac{\kappa_{\mu_a}}{2}} \\
&\quad \times \left(\frac{q}{t}\right)^{\frac{|\rho_a| + |\nu_a| - |\mu_a|}{2}} \left(\frac{t}{q}\right)^{\frac{|\sigma_a| + |\nu_{a+1}| - |\mu_a|}{2}} \\
&\quad \times s_{\nu_a / \rho_a} (t^{-\rho} q^{-\alpha_a}) s_{\mu_a / \rho_a} (t^{-\alpha_a^t} q^{-\rho}) s_{\nu_{a+1}^t / \sigma_a} (q^{-\rho} t^{-\beta_a}) s_{\mu_a^t / \sigma_a} (q^{-\beta_a^t} t^{-\rho}) \quad (3.21)
\end{aligned}$$

where $\nu_1 = \nu_{N+1} = \phi$. They involve the following sum

$$\begin{aligned}
&\sum_{\substack{\{\mu_a\}, \{\nu_a\} \\ \{\rho_a\}, \{\sigma_a\}}} \prod_a (-Q_{M,a})^{|\mu_a|} (-Q_{F,a})^{|\nu_a|} \\
&\quad \times s_{\nu_a / \rho_a} (t^{-\rho + \frac{1}{2}} q^{-\alpha_a}) s_{\mu_a / \rho_a} (t^{-\alpha_a^t} q^{-\rho - \frac{1}{2}}) s_{\nu_{a+1}^t / \sigma_a} (q^{-\rho} t^{-\beta_a - \frac{1}{2}}) s_{\mu_a^t / \sigma_a} (q^{-\beta_a^t + \frac{1}{2}} t^{-\rho})
\end{aligned}$$

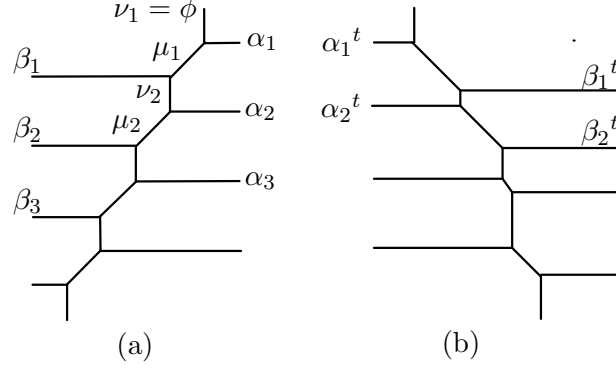


Figure 6: The building blocks of the toric Calabi-Yau that engineers $SU(N)$ gauge theory with $N_f = 2N$

Using the method of Iqbal-Kashani-Poor[27], we can take the summation. The only difference from the result of [27] is the arguments of Schur functions. Bewaring the difference, we get the sum as follows

$$\begin{aligned}
& \prod_{1 \leq a \leq b \leq N} \left[t^{-\alpha_a^t} q^{-\rho - \frac{1}{2}}, -Q_{\alpha_a \beta_b} q^{-\beta_b^t + \frac{1}{2}} t^{-\rho} \right] \prod_{1 \leq a < b \leq N} \left[q^{-\rho} t^{-\beta_a - \frac{1}{2}}, -Q_{\beta_a \alpha_b} t^{-\rho + \frac{1}{2}} q^{-\alpha_b} \right] \\
& \times \prod_{1 \leq a < b \leq N} \left\{ t^{-\alpha_a^t} q^{-\rho - \frac{1}{2}}, Q_{\alpha_a \alpha_b} t^{-\rho + \frac{1}{2}} q^{-\alpha_b} \right\} \left\{ q^{-\rho} t^{-\beta_a - \frac{1}{2}}, Q_{\beta_a \beta_b} t^{-\rho} q^{-\beta_b^t + \frac{1}{2}} \right\} \quad (3.22)
\end{aligned}$$

We provide the direct proof in appendix B. Here Kähler parameters are given by

$$\begin{aligned}
Q_{\alpha_a \beta_b} &= Q_{M,a} Q_{F,a+1} \cdots Q_{M,b-1} Q_{F,b} Q_{M,b} = Q_{a,b} Q_{M,b} \\
Q_{\beta_a \alpha_b} &= Q_{F,a+1} \cdots Q_{M,b-1} Q_{F,b} = Q_{M,a}^{-1} Q_{a,b} \\
Q_{\alpha_a \alpha_b} &= Q_{M,a} Q_{F,a+1} \cdots Q_{M,b-1} Q_{F,b} = Q_{a,b} \\
Q_{\beta_a \beta_b} &= Q_{F,a+1} \cdots Q_{M,b-1} Q_{F,b} Q_{M,b} = Q_{M,a}^{-1} Q_{a,b} Q_{M,b}
\end{aligned}$$

and we introduce

$$[x, y] \equiv \prod_{i,j=1}^{\infty} (1 + x_i y_j), \{x, y\} \equiv \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} \quad (3.23)$$

Then we obtain the following expression

$$\begin{aligned}
K_{\beta_1 \beta_2 \cdots}^{\alpha_1 \alpha_2 \cdots} &= \prod_a \left[q^{\frac{\|\alpha_a\|^2}{2}} t^{\frac{\|\beta_a\|^2}{2}} \tilde{Z}_{\alpha_a}(t, q) \tilde{Z}_{\beta_a}(q, t) \right] \\
&\times \prod_{i,j=1}^{\infty} \prod_{1 \leq a \leq b \leq N} \left(1 - Q_{\alpha_a \beta_b} t^{-\alpha_{a,i}^t + j - \frac{1}{2}} q^{-\beta_{b,j}^t + i - \frac{1}{2}} \right) \prod_{1 \leq a < b \leq N} \left(1 - Q_{\beta_a \alpha_b} t^{-\beta_{a,i} - j - \frac{1}{2}} q^{-\alpha_{b,j} + i - \frac{1}{2}} \right) \\
&\times \prod_{1 \leq a < b \leq N} \left(1 - Q_{\alpha_a \alpha_b} t^{-\alpha_{a,i}^t + j} q^{-\alpha_{b,j} + i - 1} \right)^{-1} \left(1 - Q_{\beta_a \beta_b} t^{-\beta_{a,i} + j - 1} q^{-\beta_{b,j}^t + i} \right)^{-1} \quad (3.24)
\end{aligned}$$

The amplitude for the pair of this strip geomerty Fig.6(b) is given by

$$\begin{aligned}
\tilde{K}_{\alpha_1^t \alpha_2^t \dots}^{\beta_1^t \beta_2^t \dots} &= \prod_a \left[t^{\frac{\|\alpha_a^t\|^2}{2}} q^{\frac{\|\beta_a^t\|^2}{2}} \tilde{Z}_{\alpha_a^t}(q, t) \tilde{Z}_{\beta_a^t}(t, q) \right] \\
&\times \prod_{i,j=1}^{\infty} \prod_{1 \leq a \leq b \leq N} \left(1 - Q'_{\alpha_a \beta_b} t^{-\alpha_{a,i}^t + j - \frac{1}{2}} q^{-\beta_{b,j}^t + i - \frac{1}{2}} \right) \prod_{1 \leq a < b \leq N} \left(1 - Q'_{\beta_a \alpha_b} t^{-\beta_{a,i} + j - \frac{1}{2}} q^{-\alpha_{b,j} + i - \frac{1}{2}} \right) \\
&\times \prod_{1 \leq a < b \leq N} \left(1 - Q'_{\alpha_a \alpha_b} t^{-\alpha_{a,i}^t + j - 1} q^{-\alpha_{b,j} + i} \right)^{-1} \left(1 - Q'_{\beta_a \beta_b} t^{-\beta_{a,i} + j} q^{-\beta_{b,j}^t + i - 1} \right)^{-1} \quad (3.25)
\end{aligned}$$

Gluing them, we get the Nekrasov's partition functioin for $\mathcal{N}=2$ $SU(N)$ gauge theory with $N_f = 2N$

$$Z = \sum_{\alpha_1 \alpha_2 \dots} \prod_{a=1}^N \left(f_{\alpha_a}(t, q) Q_B^{|\alpha_a|} \right) K_{\phi\phi\dots\dots}^{\alpha_1 \alpha_2 \dots} (Q_{ab}, Q_{M,a}) \tilde{K}_{\alpha_1^t \alpha_2^t \dots}^{\phi\phi\dots} (Q_{ab}, Q'_{M,a})$$

It is not so hard to generarize the above caluculation of the refined vertex for another strip geometries which contain $(-1, -1)$ curves and $(-2, 0)$ curves. Then we can engineer the Nekrasov's partition functioins for various $\mathcal{N}=2$ $SU(N)$ quiver gauge theories with matters by gluing these amplituses.

4 Conclusion

In this paper, we have applied refined topological vertex for $SU(N)$ geometries and reproduced the K-theoretic version of the Nekrasov's partition functions. From this results we can adopt refined topological vertex as a 2-parameter extension of topological A-model. We have also discussed a refined vertex on a strip geometry. Many of the nice properties obtained in [27] are maintained in the case of refined vertex. The important point is that refined vertex on strip reduces to a summation of Schur functions which is essentially discussed in [27]. Hence Schur functions of the partition functions can be summed up as in the case of the topological vertex on strips.

Acknowledgements

We would like to thank Tohru Eguchi, Yosuke Imamura, Hiroaki Kanno and Yuji Tachikawa for valuable discussions and helpful comments.

A Young diagrams and Schur functions

Young diagrams

The Young diagrams is defined as a sequence of decreasing non-negative integers

$$\mu = \{\mu_i \in \mathbb{Z}_{\geq 0} \mid \mu_1 \geq \mu_2 \geq \cdots\} \quad (\text{A.1})$$

The transpose of μ is defined as follows

$$\mu^t = \{\mu_j^t \in \mathbb{Z}_{\geq 0} \mid \mu_j^t = \#\{i \mid \mu_i \geq j\}\} \quad (\text{A.2})$$

The size and the norm of the partition is denoted as

$$|\mu| = \sum_{i=1}^{d(\mu)} \mu_i, \quad \|\mu\|^2 = \sum_{i=1}^{d(\mu)} \mu_i^2 \quad (\text{A.3})$$

For $(i, j) \in \mu$, we define the following quantities,

$$a_\mu(i, j) = \mu_i - j, \quad l_\mu(i, j) = \mu_j^t - i$$

$$a'_\mu(i, j) = j - 1, \quad l'_\mu(i, j) = i - 1$$

We introduce the hook length of the Young diagram

$$h_\mu(i, j) = \mu_i - j + \mu_j^t - i + 1$$

It is also useful to define the following quantities

$$n(\mu) = \sum_{i=1}^{d(\mu)} (i-1) \mu_i \quad \kappa_\mu = \sum_{(i,j) \in \mu} (j-i)$$

It is easy to show that they satisfy the following identities

$$n(\mu) = \frac{1}{2} \sum_{j=1}^{\mu_1} \mu_j^t (\mu_j^t - 1) = \sum_{s \in \mu} l'_\mu(s) = \sum_{s \in \mu} l_\mu(s) \quad (\text{A.4})$$

$$n(\mu^t) = \frac{1}{2} \sum_{i=1}^{d(\mu)} \mu_i (\mu_i - 1) = \sum_{s \in \mu} a'_\mu(s) = \sum_{s \in \mu} a_\mu(s) \quad (\text{A.5})$$

$$\kappa_\mu = 2(n(\mu^t) - n(\mu)) = \|\mu\|^2 - \|\mu^t\|^2 \quad (\text{A.6})$$

$$\sum_{s \in \mu} h_\mu(s) = n(\mu) + n(\mu^t) + |\mu| \quad (\text{A.7})$$

Schur functions

The Schur functions for N variables (x_1, \dots, x_N) are defined by the determinant formula

$$s_\mu(x_1, \dots, x_N) = \frac{\det_{i,j=1,\dots,N} (x_i^{\mu_j + N - j})}{\det_{i,j=1,\dots,N} (x_i^{N - j})} \quad (\text{A.8})$$

From the definition, the Schur functions are symmetric under the permutation of the variables. Moreover it is known that they form an orthogonal basis of the symmetric polynomials. We can also define the skew Schur functions by

$$s_{\mu/\nu}(x) = \sum_{\rho} c_{\nu\rho}^{\mu} s_{\rho}(x) \quad (\text{A.9})$$

Here we introduce the Richardson-Littlewood coefficients $c_{\mu\nu}^{\rho}$

$$s_{\mu}(x) s_{\nu}(x) = \sum_{\rho} c_{\mu\nu}^{\rho} s_{\rho}(x) \quad (\text{A.10})$$

We have the product expression for the Schur function of the variables $\{q^{\rho}\} = \{q^{-i+\frac{1}{2}}\}_{i=1,2,\dots}$ [30]

$$s_{\mu}(q^{-\rho}) \equiv s_{\mu}(q^{\frac{1}{2}}, q^{\frac{3}{2}}, \dots) = q^{\frac{\|\mu^t\|^2}{2}} \tilde{Z}_{\mu}(q) \quad (\text{A.11})$$

where

$$\tilde{Z}_{\mu}(q) = \prod_{s \in \mu} \left(1 - q^{h_{\mu}(s)}\right)^{-1} \quad (\text{A.12})$$

Using this formula, we obtain

$$s_{\mu}(q^{\rho}) = q^{\frac{\kappa_{\mu}}{2}} s_{\mu^t}(q^{\rho}) = (-1)^{|\mu|} s_{\mu^t}(q^{-\rho}) \quad (\text{A.13})$$

Let us introduce the 2-parameter extension of $\tilde{Z}_{\mu}(q)$ by

$$P_{\mu}(t^{-\rho}; q, t) = t^{\frac{\|\mu^t\|^2}{2}} \tilde{Z}_{\mu^t}(t, q) \quad (\text{A.14})$$

$$\tilde{Z}_{\mu}(t, q) = \prod_{s \in \mu} \left(1 - t^{a_{\mu}(s)+1} q^{l_{\mu}(s)}\right)^{-1} \quad (\text{A.15})$$

It appears in the refinement of topological vertex:

$$C_{\phi\phi\mu}(q) = q^{\frac{\|\mu\|^2}{2}} \tilde{Z}_{\mu}(q) \rightarrow C_{\phi\phi\mu}(t, q) = q^{\frac{\|\mu\|^2}{2}} \tilde{Z}_{\mu}(t, q) \quad (\text{A.16})$$

In summing the Schur functions, we use the following identities

$$\sum_{\mu} s_{\mu}(x) s_{\mu}(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \quad (\text{A.17})$$

$$\sum_{\mu} s_{\mu^t}(x) s_{\mu}(y) = \prod_{i,j} (1 + x_i y_j) \quad (\text{A.18})$$

$$\sum_{\mu} s_{\mu/\rho}(x) s_{\mu/\sigma}(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\nu} s_{\rho/\nu}(y) s_{\sigma/\nu}(x) \quad (\text{A.19})$$

$$\sum_{\mu} s_{\mu^t/\rho}(x) s_{\mu/\sigma}(y) = \prod_{i,j} (1 + x_i y_j) \sum_{\nu} s_{\rho/\nu^t}(y) s_{\sigma^t/\nu^t}(x) \quad (\text{A.20})$$

$$s_{\mu}(Qx) = Q^{|\mu|} s_{\mu}(x) \quad (\text{A.21})$$

$$s_{\mu/\nu}(Qx) = Q^{|\mu|-|\nu|} s_{\mu/\nu}(x) \quad (\text{A.22})$$

They are important identities which we use throughout the paper.

B Proof of Formula

In this appendix, we prove the following identity for section 3.

$$\begin{aligned} & \prod_{a=1}^N Q_{M,a}^{|\mu_a|} \prod_{a=2}^N Q_{F,a}^{|\nu_a|} \sum_{\substack{\{\mu_i\}, \{\nu_i\} \\ \{\rho_i\}, \{\sigma_i\}}} \prod_{a=1}^N s_{\nu_a/\rho_a}(w^{(a)}) s_{\mu_a/\rho_a}(x^{(a)}) s_{\mu_a^t/\sigma_{a+1}}(y^{(a)}) s_{\nu_{a+1}^t/\sigma_{a+1}}(z^{(a+1)}) \\ &= \prod_{1 \leq a \leq b \leq N} \left[x^{(a)}, Q_{\alpha_a \beta_b} y^{(b)} \right] \prod_{1 \leq a < b \leq N} \left[z^{(a+1)}, Q_{\beta_a \alpha_b} w^{(b)} \right] \left\{ x^{(a)}, Q_{\alpha_a \alpha_b} w^{(b)} \right\} \left\{ z^{(a+1)}, Q_{\beta_a \beta_b} y^{(b)} \right\} \end{aligned} \quad (\text{B.1})$$

where we take the sum over the Young diagrams $\mu_1 \cdots \mu_N$, $\nu_2 \cdots \nu_N$, $\rho_2 \cdots \rho_N$, and $\sigma_2 \cdots \sigma_N$. Notice that we denote $\rho_1 = \sigma_{N+1} = \phi$. in the formula.

Let us show the identity. The first line of this equation becomes

$$\begin{aligned} & \sum_{\substack{\rho_2 \cdots \rho_N \\ \sigma_2 \cdots \sigma_N}} \prod_{a=2}^N Q_{M,a}^{|\rho_a|} Q_{F,a}^{|\sigma_a|} \\ & \times \sum_{\substack{\mu_1 \cdots \mu_N \\ \nu_2 \cdots \nu_N}} \prod_{a=1}^N s_{\mu_a/\rho_a}(Q_{M,a} x^{(a)}) s_{\mu_a^t/\sigma_{a+1}}(y^{(a)}) s_{\nu_{a+1}^t/\sigma_{a+1}}(Q_{F,a+1} z^{(a+1)}) s_{\nu_{a+1}/\rho_{a+1}}(w^{(a+1)}) \\ &= \prod_{a=1}^N \left[x^{(a)}, Q_{M,a} y^{(a)} \right] \prod_{a=2}^N \left[z^{(a+1)}, Q_{F,a+1} w^{(a+1)} \right] \sum_{\substack{\alpha_2 \cdots \alpha_{N-1} \\ \beta_2 \cdots \beta_N}} \sum_{\substack{\rho_2 \cdots \rho_N \\ \sigma_2 \cdots \sigma_N}} \prod_{a=2}^N Q_{M,a}^{|\alpha_a|} Q_{F,a}^{|\beta_a|} \\ & \times \prod_{a=1}^N s_{\sigma_{a+1}^t/\alpha_a}(Q_{M,a} x^{(a)}) s_{\rho_a^t/\alpha_a}(Q_{M,a} y^{(a)}) s_{\rho_{a+1}^t/\beta_{a+1}}(Q_{F,a+1} z^{(a+1)}) s_{\sigma_{a+1}^t/\beta_{a+1}}(Q_{F,a+1} w^{(a+1)}) \end{aligned}$$

$$\begin{aligned}
&= \prod_{a=1}^N \left[x^{(a)}, Q_{M,a} y^{(a)} \right] \prod_{a=2}^{N-1} \left[z^{(a)}, Q_{F,a} w^{(a)} \right] \prod_{a=1}^N \left\{ x^{(a)}, Q_{M,a} Q_{F,a+1} w^{(a+1)} \right\} \left\{ z^{(a+1)}, Q_{F,a+1} Q_{M,a+1} y^{(a+1)} \right\} \\
&\times \sum_{\substack{\beta_1 \cdots \beta_{N-1} \\ \alpha_2 \cdots \alpha_{N-1}}} \sum_{\substack{\gamma_2 \cdots \gamma_{N-1} \\ \delta_2 \cdots \delta_{N-1}}} \prod_{a=1}^{N-1} Q_{F,a+1}^{|\beta_a|} \prod_{a=2}^{N-1} Q_{M,a}^{|\alpha_a|} \\
&\times \prod_{a=1}^{N-1} s_{\alpha_a / \gamma_a} (Q_{F,a+1} w^{(a+1)}) s_{\beta_a / \gamma_a} (Q_{M,a} x^{(a)}) s_{\beta_a t / \delta_{a+1}} (Q_{M,a+1} y^{(a+1)}) s_{\alpha_{a+1} t / \delta_{a+1}} (Q_{F,a+1} z^{(a+1)})
\end{aligned}$$

Using this result repeatedly, we obtain the second line of the formula (B.1).

References

- [1] For a review see, e.g., M. Marino, “Chern-Simons theory and topological strings,” *Rev. Mod. Phys.* **77**, 675 (2005) [arXiv:hep-th/0406005]. M. Marino, “Les Houches lectures on matrix models and topological strings,” arXiv:hep-th/0410165. A. Neitzke and C. Vafa, “Topological strings and their physical applications,” arXiv:hep-th/0410178. and references therein.
- [2] R. Gopakumar and C. Vafa, “Topological gravity as large N topological gauge theory,” *Adv. Theor. Math. Phys.* **2**, 413 (1998) [arXiv:hep-th/9802016].
- [3] R. Gopakumar and C. Vafa, “M-theory and topological strings. I,” arXiv:hep-th/9809187.
- [4] R. Gopakumar and C. Vafa, “On the gauge theory/geometry correspondence,” *Adv. Theor. Math. Phys.* **3**, 1415 (1999) [arXiv:hep-th/9811131].
- [5] M. Aganagic, A. Klemm, M. Marino and C. Vafa, “The topological vertex,” *Commun. Math. Phys.* **254**, 425 (2005) [arXiv:hep-th/0305132].
- [6] S. H. Katz and C. Vafa, “Geometric engineering of N = 1 quantum field theories,” *Nucl. Phys. B* **497**, 196 (1997) [arXiv:hep-th/9611090]. S. Katz, P. Mayr and C. Vafa, “Mirror symmetry and exact solution of 4D N = 2 gauge theories. I,” *Adv. Theor. Math. Phys.* **1**, 53 (1998) [arXiv:hep-th/9706110].
- [7] N. A. Nekrasov, “Seiberg-Witten prepotential from instanton counting,” *Adv. Theor. Math. Phys.* **7**, 831 (2004) [arXiv:hep-th/0206161].
- [8] A. Iqbal and A. K. Kashani-Poor, “Instanton counting and Chern-Simons theory,” *Adv. Theor. Math. Phys.* **7**, 457 (2004) [arXiv:hep-th/0212279].

- [9] A. Iqbal and A. K. Kashani-Poor, “SU(N) geometries and topological string amplitudes,” *Adv. Theor. Math. Phys.* **10**, 1 (2006) [arXiv:hep-th/0306032].
- [10] T. Eguchi and H. Kanno, “Topological strings and Nekrasov’s formulas,” *JHEP* **0312**, 006 (2003) [arXiv:hep-th/0310235].
- [11] T. J. Hollowood, A. Iqbal and C. Vafa, “Matrix Models, Geometric Engineering and Elliptic Genera,” arXiv:hep-th/0310272.
- [12] J. Zhou, “Curve counting and instanton counting,” arXiv:math.AG/0311237.
- [13] T. Eguchi and H. Kanno, “Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes,” *Phys. Lett. B* **585**, 163 (2004) [arXiv:hep-th/0312234].
- [14] R. Flume and R. Poghossian, “An algorithm for the microscopic evaluation of the coefficients of the Seiberg-Witten prepotential,” *Int. J. Mod. Phys. A* **18**, 2541 (2003) [arXiv:hep-th/0208176].
- [15] U. Bruzzo, F. Fucito, J. F. Morales and A. Tanzini, “Multi-instanton calculus and equivariant cohomology,” *JHEP* **0305**, 054 (2003) [arXiv:hep-th/0211108].
- [16] H. Awata and H. Kanno, “Instanton counting, Macdonald functions and the moduli space of D-branes,” *JHEP* **0505**, 039 (2005) [arXiv:hep-th/0502061].
- [17] J. Zhou, “On a deformed topological vertex,” arXiv:math.AG/0504460.
- [18] A. Iqbal, C. Kozcaz and C. Vafa, “The refined topological vertex,” arXiv:hep-th/0701156.
- [19] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, “Topological amplitudes in string theory,” *Nucl. Phys. B* **413**, 162 (1994) [arXiv:hep-th/9307158].
- [20] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” *Commun. Math. Phys.* **165**, 311 (1994) [arXiv:hep-th/9309140].
- [21] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” *Nucl. Phys. B* **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087].
- [22] R. Gopakumar and C. Vafa, “M-theory and topological strings. II,” arXiv:hep-th/9812127.
- [23] N. Nekrasov and A. Okounkov, “Seiberg-Witten theory and random partitions,” arXiv:hep-th/0306238.

- [24] H. Nakajima and K. Yoshioka, “Instanton counting on blowup. I. 4-dimensional pure gauge theory,” *Invent. Math* **162**, no. 2, 313 (2005) [arXiv:math.A.G/0306198].
- [25] A. Braverman and P. Etingof, “Instanton counting via affine Lie algebras II: from Whittaker vectors to the Seiberg-Witten prepotential” arXiv:math.AG/0409441.
- [26] H. Nakajima, “Lectures on Hilbert Schemes of Points on Surfaces”, American Mathematical Society, University Lectures Series v.18 (1999)
- [27] A. Iqbal and A. K. Kashani-Poor, “The vertex on a strip,” *Adv. Theor. Math. Phys.* **10**, 317 (2006) [arXiv:hep-th/0410174].
- [28] Y. Tachikawa, “Five-dimensional Chern-Simons terms and Nekrasov’s instanton counting,” *JHEP* **0402**, 050 (2004) [arXiv:hep-th/0401184].
- [29] L. Gottsche, H. Nakajima and K. Yoshioka, “K-theoretic Donaldson invariants via instanton counting,” arXiv:math/0611945.
- [30] I. G. Macdonald, “Symmetric functions and Hall polynomials,” (second edition, 1995), Oxford Mathematical Monographs, Oxford Science Publications.